Exact solutions of the derivative nonlinear Schrödinger equation for a nonlinear transmission line

E. Kengne[1,2,*] and W. M. Liu[2,†]
[1]Department of Mathematics & Computer Science, Faculty of Science, University of Dschang, P. O. Box 4509, Douala, Republic of Cameroon
[2]Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100080, China
(Received 4 October 2005; revised manuscript received 10 November 2005; published 6 February 2006)

We consider the derivative nonlinear Schrödinger equation with constant potential as a model for wave propagation on a discrete nonlinear transmission line. This equation can be derived in the small amplitude and long wavelength limit using the standard reductive perturbation method and complex expansion. We construct some exact soliton and elliptic solutions of the mentioned equation by perturbation of its Stokes wave solutions. We find that for some values of the coefficients of the equation and for some parameters of solutions, the graphical representations show some kinds of symmetries such as mirror symmetry and rotational symmetry.

DOI: 10.1103/PhysRevE.73.026603 PACS number(s): 42.65.Tg, 42.25.Bs, 84.40.Az

I. INTRODUCTION

Over the past 35 years the concept of solitons has entered into various branches of mathematics and physics. Solitons are stationary pulses or wave packets which propagate in nonlinear dispersive media. They keep their stable wave forms due to a dynamical balance between the nonlinear and the dispersive effects. The first experimental observations of solitons were made in 1834 by Russell [1], but it was only after the computer simulations of Zabusky and Kruskal that the present interest in solitons grew [2]. During the subsequent 30 years our knowledge of solitons has developed to a mature theory. Many soliton equations describing nonlinear systems are known and solitons themselves have been observed (directly or indirectly) in various media. However, there are only a few systems where solitons are easily and directly observed in controlled laboratory experiments. Nonlinear electrical transmission lines are good examples of such systems [3,4].

A nonlinear transmission line is comprised of a transmission line periodically loaded with varactors, where the capacitance nonlinearity arises from the variable depletion layer width, which depends both on the dc bias voltage and on the ac voltage of the propagating wave. For example, the model shown in Fig. 1 is a lossless discrete nonlinear transmission line made of ladder-type LC circuits containing constant inductors and voltage-dependent capacitors. Each cell contains a linear inductance L_1 in parallel with a linear capacitance C_S in the series branch and, a linear inductance L_2 in parallel with a nonlinear capacitance $C(V)$ in the shunt branches. The nonlinear capacitors are usually reverse-biased capacitance diodes. In typical experiments the number of identical sections is between 50 and 1000.

Using reductive perturbation method and complex expansion for the nonlinear transmission line shown on Fig. 1, one easily derives the following version of the complex Ginzburg-Landau equation (see [5–8]):

$$i\frac{\partial u}{\partial t} = P\frac{\partial^2 u}{\partial x^2} + \gamma u + Q_1|u|^4 u + iQ_2|u|^2\frac{\partial u}{\partial x} + iQ_3 u^2\frac{\partial u^*}{\partial x}, \quad (1)$$

where the coefficients P, Q_1, Q_2, Q_3, and γ are real and are given in terms of the transmission line's parameters. Equation (1) is called the derivative nonlinear Schrödinger (DNLS) equation with potential, or the cubic-quintic Ginzburg-Landau equation.

The amplitude equations of cubic-quintic Ginzburg-Landau type (1) have been derived via asymptotic analysis of the governing (Navier-Stokes) equations of fluid mechanics in the limit of long spatial and slow temporal oscillations near the onset of instability [9]. The necessity to include the higher order terms of (1) in some physical contexts (e.g., binary fluid convection) was partly explained or numerically checked by Schof and Zimmermann [10,11], Glazier and Kolodner [12], Brand and Deissler [13], and Deissler and Brand [14]. The two terms $u^2\partial u^*/\partial x$ and $|u|^2\partial u/\partial x$, called derivative nonlinear terms, appear naturally in the asymptotic derivation. Deissler and Brand [14] showed numerically that these two terms can significantly slow down the propagating speed of the pulses and also cause the nonsymmetry of pulses. This is interesting in view of the drifting pulse states observed experimentally by Glazier and Kolodner [12], for example.

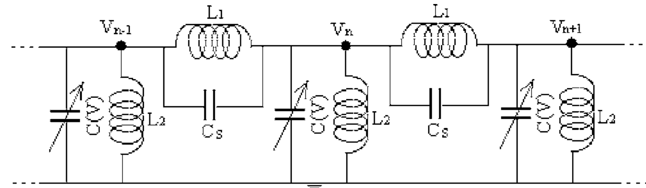

FIG. 1. Nonlinear transmission line, where V_n denotes ac voltage and I_n and I_{n+1} ac currents.

*Corresponding author. Electronic address: ekengne6@yahoo.fr
†Electronic address: wmliu@aphy.iphy.ac.cn

The derivative nonlinear Schrödinger equation (1) is used for modeling of wave processes in different physical systems such as nonlinear optics [15–17], circular polarized Alfvéen waves in plasma [18,19], Stokes waves in fluids of finite depth, etc. In nonlinear optics [15–17], Eq. (1) can be derived in a systematic way by means of the reductive perturbation scheme as a model for single mode propagation. In the context of waveguides as optical fibers, t usually corresponds to the propagation distance of the electric field envelope u of an optical beam along the fiber, x plays the role of the time, the term $Q_1|u|^4u$ models the nonlinear Kerr effect, while $Q_2|u|^2\partial u/\partial x$ and $Q_3u^2\partial u^*/\partial x$ are the nonlinear dispersion contributions.

There are many partial cases of Eq. (1). For example, when the derivative nonlinear contributions are neglected, Eq. (1) becomes

$$i\frac{\partial u}{\partial t} = P\frac{\partial^2 u}{\partial x^2} + \gamma u + Q_1|u|^4 u, \qquad (2)$$

which is a quintic defocusing Schrödinger equation with potential. Semilinear Schrödinger equations—with and without potentials, and with various nonlinearities—arise as models for diverse physical phenomena, including Bose-Einstein condensates and as a description of the envelope dynamics of a general dispersive wave in a weakly nonlinear medium. In the theory of Bose-Einstein condensates, the coefficient γ in Eq. (2) is generally a function of the spatial variable x and is called the potential function, while the coefficient Q_1 in this equation corresponds to elastic collisions within the condensates [20–25].

In this paper, we deal with the derivative nonlinear Schrödinger equation (1) $(\Sigma_{j=1}^{3}|Q_j|>0)$ and assume that it models wave processes not only on the nonlinear transmission line presented in Fig. 1, but also in other different physical systems, such as nonlinear optics, for example. The structure of the rest of the paper is the following. In Sec. II, we derive exact soliton solutions for Eq. (1) by perturbing the plane wave solutions of Eq. (1). In Sec. III, we present some nonsolitary solutions of Eq. (1), some of which at a given limit give other soliton solutions. Here we show that some nonsolitary solutions of the DNLS Eq. (1) present some kinds of symmetries.

II. SOLITON SOLUTIONS FOR DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

In this section we show that the DNLS equation (1) admits Stokes solutions that one can perturb to obtain soliton solutions. First, we express $u(x,t)$ in polar form,

$$u(x,t) = a(x,t)\exp[i\varphi(x,t)], \qquad (3)$$

where $a(x,t)$ and $\varphi(x,t)$ are real functions. Substituting Eq. (3) into Eq. (1) and separating real and imaginary parts, we have

$$a\varphi_t + P(a_{xx} - a(\varphi_x)^2) + \gamma a + Q_1a^5 - Q_2a^3\varphi_x + Q_3a^3\varphi_x = 0,$$

$$-a_t + P(2a_x\varphi_x + a\varphi_{xx}) + Q_2a^2a_x + Q_3a^2a_x = 0. \qquad (4)$$

The system (4) admits nontrivial solutions of the form

$$a = a_0, \quad \varphi = l_0x - q_0t,$$

$$q_0 = Q_1a_0^4 - l_0a_0^2Q_2 + l_0a_0^2Q_3 - Pl_0^2 + \gamma \qquad (5)$$

where $a_0 \neq 0$ and l_0 are two real constants. If we insert the expressions for a and φ into Eq. (3), we obtain the following Stokes solutions of the DNLS equation (1): $u(x,t) = a_0\exp\{i[l_0x-(Q_1a_0^4-l_0a_0^2Q_2+l_0a_0^2Q_3-Pl_0^2+\gamma)]t\}$ with parameters a_0 and l_0.

Now, we seek a solution for the system of Eqs. (4) by pretreating the nontrivial solutions (5) as follows:

$$a(x,t) = a_0 + \alpha(x - vt = z),$$

$$\varphi(x,t) = l_0(x - vt) + \widetilde{\Psi}(z) - (q_0 - l_0v)t = \Psi(z) - (q_0 - l_0v)t, \qquad (6)$$

for some constant v. If we insert Eq. (6) into system (4), we get

$$-va\Psi' + P[\alpha'' - a(\Psi')^2] + Q_1a^5 + (Q_3 - Q_2)a^3\Psi' + (\gamma + l_0v - q_0)a = 0,$$

$$P(2\alpha'\Psi' + a\Psi'') + (Q_2 + Q_3)a^2\alpha' + v\alpha' = 0. \qquad (7)$$

Multiplying the second equation of system (7) by a and integrating the resulting equation yields

$$\Psi' = \frac{K_1}{Pa^2} - \frac{Q_2 + Q_3}{4P}a^2 - \frac{v}{2P}, \qquad (8)$$

where K_1 is a constant of integration. Inserting Eq. (8) into the first equation (7) yields

$$a'' = \frac{1}{4P^2}[4K_1(Q_2 - Q_3) - 2K_1(Q_2 + Q_3) - v^2 - 4P(\gamma + l_0v - q_0)]a + \frac{v(Q_3 - Q_2)}{2P^2}a^3 + \frac{K_1^2}{P^2a^3} + \frac{(Q_2 + Q_3)(5Q_3 - 3Q_2) - 16PQ_1}{4P^2}a^5.$$

A first integral of this last equation is

$$a'^2 = \frac{1}{4P^2}[4K_1(Q_2 - Q_3) - 2K_1(Q_2 + Q_3) - v^2 - 4P(\gamma + l_0v - q_0)]a^2 + \frac{v(Q_3 - Q_2)}{4P^2}a^4 + \frac{K_2}{4} + \frac{(Q_2 + Q_3)(5Q_3 - 3Q_2) - 16PQ_1}{12P^2}a^6 - \frac{K_1^2}{P^2a^2},$$

where K_2 is a constant of integration. By setting $a^2=\zeta$, we obtain

$$\zeta'^2 = -\frac{4K_1^2}{P^2} + K_2\zeta + C\zeta^2 + D\zeta^3 + E\zeta^4, \qquad (9)$$

where

$$C = \frac{2K_1(Q_2 - 3Q_3) - 4P(\gamma + l_0v - q_0) - v^2}{P^2},$$

$$D = \frac{v(Q_3 - Q_2)}{P^2},$$

$$E = \frac{(Q_2 + Q_3)(5Q_3 - 3Q_2) - 16PQ_1}{3P^2}. \qquad (10)$$

Equation (9) is an elliptic ordinary differential equation. In some special cases, we find its particular positive bounded solutions.

A. Case $K_1 = K_2 = 0$

When $K_1 = K_2 = 0$, Eq. (9) becomes $\zeta'^2/\zeta^2 = C + D\zeta + E\zeta^2$. The case $K_2 = 0$ corresponds to solutions that decrease to zero at infinity. If coefficients C, D, and E of Eq. (10) satisfy the conditions

$$C > 0 \quad \text{and} \quad D^2 - 4CE > 0,$$

then it admits soliton solutions

$$\zeta(z) = \zeta^{\pm}(z) = \frac{2C}{-D \pm \sqrt{D^2 - 4CE}\cosh(\sqrt{C}z)}.$$

Notice that from Eqs. (10) we have

$$C = \frac{-4P(\gamma + l_0 v - q_0) - v^2}{P^2} > 0$$

$$\Leftrightarrow v^2 + 4P(\gamma + l_0 v - q_0) < 0.$$

The sign $-$ or $+$ is taken from the condition of positivity of $\zeta(z)$.

B. Case $E = 0$ and $D \neq 0$

In the case when $E = 0$ and $D \neq 0$, Eq. (9) reads

$$\zeta'^2 = -\frac{4K_1^2}{P^2} + K_2\zeta + C\zeta^2 + D\zeta^3. \qquad (11)$$

Because $\Sigma_{j=1}^3 |Q_j| > 0$, we have $D \neq 0$. If the two constants of integration K_1 and K_2 are taken from conditions that

$$0 < C^2 - 3DK_2$$

and

$$K_1^2 = \frac{1}{108D^2}[9DP^2K_2(-C\sqrt{C^2 - 3DK_2}) + 3P^2C(-C\sqrt{C^2 - 3DK_2})^2 + P^2(-C\sqrt{C^2 - 3DK_2})^3],$$

then Eq. (11) admits the following soliton solution:

$$\zeta(z) = \frac{(-C + \sqrt{C^2 - 3DK_2})}{3D} - \frac{2C + (1 + D)(-C + \sqrt{C^2 - 3DK_2})}{2D\cosh^2\left(\frac{(C^2 - 3DK_2)^{1/4}}{2}z\right)}. \qquad (12)$$

For given coefficients P, Q_1, Q_2, Q_3, and γ of Eq. (1) satisfying conditions $E = 0$ and $D \neq 0$, the solution class (12) has one free parameter K_2 which plays the role of a constant background level or offset. The freedom in choosing a_0, l_0, and v gives a total of four free parameters K_2, a_0, l_0, and v. The requirements that $C^2 - 3DK_2 > 0$ and that both $\zeta(z)$ and K_1^2 are positive impose conditions on the domain of these parameters.

III. SOME ELLIPTIC SOLUTIONS AND OTHER SOLITON SOLUTIONS OF EQUATION (1)

In this section we build some nonsoliton solutions of the quintic Ginzburg-Landau equation (1) from where we derive some other solitary solutions. Let us come back to Eq. (9):

$$\zeta'^2 = -\frac{4K_1^2}{P^2} + K_2\zeta + C\zeta^2 + D\zeta^3 + E\zeta^4. \qquad (13)$$

The following three subsections describe two classes of bounded nonnegative solutions of this equation.

A. Type A: $E = 0$

For this case, Eq. (13) becomes

$$\zeta'^2 = -\frac{4K_1^2}{P^2} + K_2\zeta + C\zeta^2 + D\zeta^3, \qquad (14)$$

and solution $\zeta(z)$ is a quadratic function of either $\mathrm{cn}(z,k)$

$$\zeta(z) = A_0 + A_1\mathrm{cn}^2(\mu z, k) \qquad (15)$$

or $\mathrm{sn}(x,k)$

$$\zeta(z) = \widetilde{A}_0 + \widetilde{A}_1\mathrm{sn}^2(\mu z, k), \qquad (16)$$

where $\mathrm{cn}(x,k)$ and $\mathrm{sn}(x,k)$ are the Jacobi elliptic cosine and sine, respectively, with modulus $k \in [0,1]$ (see Ref. [26–30]). Inserting the ansatz (15) in Eq. (14) and equating the coefficients of equal powers of $\mathrm{cn}(\mu z,k)$ results in relations among the solution parameters a_0, l_0, v, $\mu \neq 0$, k, K_1, and K_2, and the equation coefficients P, γ, and Q_j, $j = 1,2,3$. These are

$$A_0 = \frac{4\mu^2(2k^2 - 1) - C}{3D}, \quad A_1 = -\frac{4k^2\mu^2}{D},$$

$$k^2 = \frac{1}{2} \pm \frac{\sqrt{C^2 - 3K_2D - 12\mu^4}}{4\mu^2}, \qquad (17)$$

and μ is any real root of the equation

$$0 = 3C(-C \pm 2\sqrt{C^2 - 3K_2D - 12\mu^4})^2 - \frac{108K_1^2D^2}{P^2} + 9K_2D(-C \pm 2\sqrt{C^2 - 3K_2D - 12\mu^4}) + (-C \pm 2\sqrt{C^2 - 3K_2D - 12\mu^4})^3 \qquad (18)$$

with

$$C^2 - 3K_2D - 12\mu^4 \geqslant 0.$$

In Eqs. (17) and (18), sign $+$ or $-$ is chosen from the condition that $0 \leqslant k \leqslant 1$. Moreover, solution $\zeta(z)$ given by Eq. (15)

must be positive. This condition of positivity of $\zeta(z)$ imposes some restrictions on the region of parameters of solution (15). For example, let us consider the case when $K_1=K_2=0$. Then $A_0=0$, $A_1=-(|C|+C)/2D$, $k^2=\frac{1}{2}(1+C/|C|)$, $\mu^2=|C|/4$. In order that $A_1\neq 0$, it is necessary and sufficient that $C>0$, and we then have $A_0=0$, $A_1=-C/D$, $k^2=1$, $\mu^2=C/4$, and Eq. (15) gives the soliton solution

$$\zeta(z)=-\frac{C}{D\cosh^2[(\sqrt{C}/2)z]}. \qquad (19)$$

For the positivity of this solution, it is necessary and sufficient that $D<0$. We then obtain the following conditions for solution (19):

$$C>0 \quad \text{and} \quad D<0.$$

If we go back to Eqs. (10), we have the following conditions on coefficients P, γ, and Q_j $(j=1,2,3)$ of Eq. (1), the parameters a_0 and l_0 of the Stokes wave solution (5), and the velocity v:

$$4P[l_0v+l_0a_0^2(Q_2-Q_3)+Pl_0^2-Q_1a_0^4]+v^2<0,$$

$$v(Q_3-Q_2)<0.$$

If we take l_0 as

$$l_0=\frac{a_0^2(Q_3-Q_2)-v}{P}, \qquad (20)$$

the first of the above conditions becomes

$$0>v^2-4PQ_1a_0^4=v^2-\frac{a_0^4(Q_2+Q_3)(5Q_3-3Q_2)}{4},$$

and a necessary condition is that $(Q_2+Q_3)(5Q_3-3Q_2)>0$. Here we have used the condition $E=0$. Hence, for the existence of solution (19) with l_0 given by Eq. (20), the parameters a_0 and v must satisfy conditions

$$\frac{v^2}{a_0^4}<\frac{(Q_2+Q_3)(5Q_3-3Q_2)}{4} \quad \text{and} \quad v(Q_3-Q_2)<0.$$

The soliton solution (19) coincides with the soliton solution (12) in which one takes $K_2=K_1=0$.

For the construction of the solution (16), we use the solution (15) and the Jacobian elliptic function identities [31]. Inserting the identity $\mathrm{sn}^2(\mu z,k)=1-cn^2(\mu z,k)$ into Eq. (16) yields

$$\zeta(z)=\widetilde{A}_0+\widetilde{A}_1-\widetilde{A}_1\mathrm{cn}^2(\mu z,k)=A_0+A_1\mathrm{cn}^2(\mu z,k).$$

Using now solution (15), we obtain

$$\widetilde{A}_0=-\frac{4\mu^2(1+k^2)+C}{3D}, \quad \widetilde{A}_1=\frac{4k^2\mu^2}{D},$$

$$k^2=\frac{1}{2}\pm\frac{\sqrt{C^2-3K_2D-12\mu^4}}{4\mu^2}, \qquad (21)$$

and μ is any real solution of Eq. (18).

We now use either solution (15) or (16), let us say, solution (16), to construct another elliptic solution of Eq. (14).

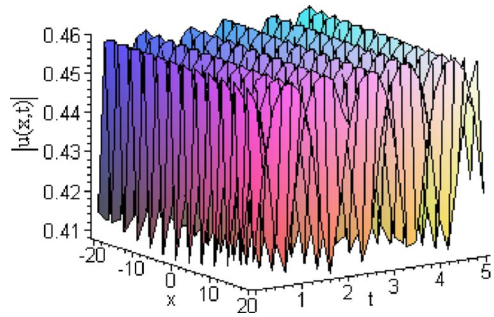

FIG. 2. (Color online) Evolution of type A $\mathrm{cn}(x,t)$ solution for $k=0.5$ and $v=0.5$ over 15 time units.

From the Jacobian elliptic function identities we have $\mathrm{sn}^2(\mu z,k)=[1-\mathrm{dn}^2(\mu z,k)]/k^2$, which in Eq. (16) gives

$$\zeta(z)=\frac{\widetilde{A}_0k^2+\widetilde{A}_1}{k^2}-\frac{\widetilde{A}_1}{k^2}\mathrm{dn}^2(\mu z,k), \qquad (22)$$

where $\mathrm{dn}(z,k)$ denotes the third Jacobian elliptic function with elliptic modulus $k\in[0,1]$. Here $\widetilde{A}_0$ and $\widetilde{A}_1$ are given by Eq. (21) and μ is any real root of Eq. (18).

Both $\mathrm{cn}(z,k)$ and $\mathrm{sn}(z,k)$ have zero average as functions of z and lie in $[-1,1]$. On the other hand, $\mathrm{dn}(z,k)$ has nonzero average; its range is $[\sqrt{1-k^2},1]$. Furthermore, $\mathrm{cn}(z,k)$ and $\mathrm{sn}(z,k)$ are periodic in z with period $4K(k)$, whereas $\mathrm{dn}(z,k)$ is periodic with period $2K(k)$. Solutions (15), (16), and (22) are shown on Figs. 2–4, respectively. We use the following data:

$$Q_3=-\frac{1}{2}, \quad P=\frac{1}{2}, \quad Q_2=\frac{5}{2}, \quad Q_1=-\frac{5}{2}. \qquad (23)$$

Equations (10) give

$$D=-12v, \quad C=-4v^2,$$

where we have taken

$$l_0=-2v-6a_0^2 \quad \text{and} \quad K_1=\frac{5}{8}a_0^4.$$

We choose μ from the condition $k=\frac{1}{2}$, which allows us to take the sign $-$ in Eqs. (17) and (18). We then obtain $\mu^4=16v^4$, if we take $K_2=\frac{16}{3}v^3$. Equation (18) then gives

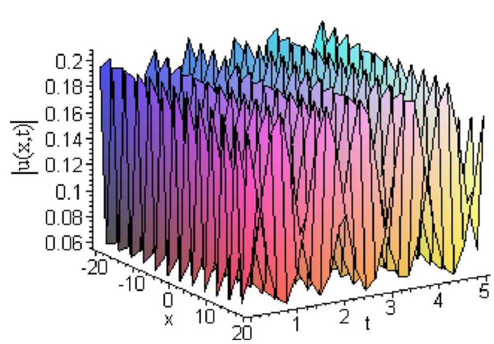

FIG. 3. (Color online) Evolution of type A $\mathrm{sn}(x,t)$ solution for $k=0.5$ and $v=\sqrt[4]{13}$ over 5 time units.

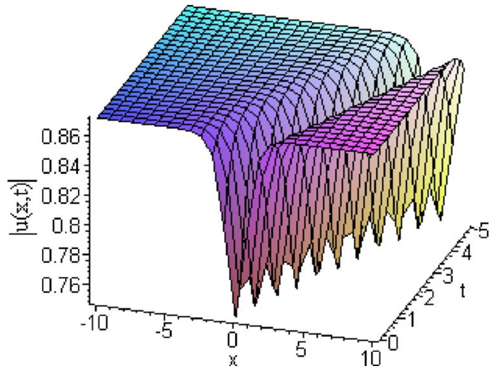

FIG. 4. (Color online) Evolution of type A $\mathrm{dn}(z,k)$ solution at limit $k\to 1$ for $v=-2^{3/4}$ over 5 time units.

$$a_0^8=\frac{64}{2025}v^4.$$

Inserting these values of parameters into Eq. (17) yields $A_0=v/9$, $A_1=v/3$, $k=1/2$. By taking $\mu=2v$, Eqs. (15) and (22) give $\zeta(z)=(v/3)[1/3+\mathrm{cn}^2(2vz,1/2)]$ and $\zeta(z)=(4v/3)\times[-2/3+\mathrm{dn}^2(2vz,1/2)]$, respectively, which are nonnegative for every $v>0$.

For solutions given by (16) and (22), we take $K_2=-\frac{15}{16}v^3$, and this gives $a_0^8=[(39m^5-78m^4-4m^3+32)/(27\times 9\times 50)]\times v^4$, with $m=1/\sqrt{13}$.

Solutions given by Eq. (16) and (22) then become

$$\zeta(z)=-\frac{v}{12\sqrt{13}}\left[\frac{3-4\sqrt{13}}{3}-\mathrm{sn}^2\left(-\frac{v}{\sqrt[4]{13}}z,\frac{1}{2}\right)\right]$$

and

$$\zeta(z)=v\left[\frac{4\sqrt{13}+9}{36\sqrt{13}}-\frac{1}{3\sqrt{13}}\mathrm{dn}^2\left(-\frac{v}{\sqrt[4]{13}}z,\frac{1}{2}\right)\right],$$

which are nonnegative for every $v>0$.

1. The hyperbolic limit

In the limit $k\to 1$, the elliptic functions reduce to hyperbolic functions. Then Eqs. (19) and (22) give

$$\zeta(z)=A_0+\frac{A_1}{\cosh^2(\mu z)},$$

$$\zeta(z)=\widetilde{A}_0+\widetilde{A}_1-\frac{\widetilde{A}_1}{\cosh^2(\mu z)}, \tag{24}$$

respectively, and Eq. (16) gives

$$\zeta(z)=\widetilde{A}_0+\widetilde{A}_1\tanh^2(\mu z). \tag{25}$$

Thus in the limit $k\to 1$, elliptic solutions (15), (24), and (16) give the soliton solutions (24) and (25), respectively. From Eq. (17), the limit $k\to 1$ gives

$$1=\pm\frac{\sqrt{C^2-3K_2D-12\mu^4}}{2\mu^2},$$

and the sign + is taken. We then have

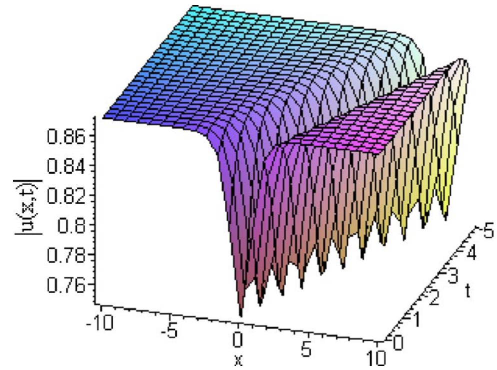

FIG. 5. (Color online) Evolution of type A $\mathrm{cn}(z,k)$ solution at limit $k\to 1$ for $v=-2^{3/4}$ over 5 time units.

$$\mu=\frac{\sqrt[4]{C^2-3K_2D}}{2},\quad A_0=\frac{\sqrt{C^2-3K_2D}-C}{3D},$$

$$A_1=-\frac{\sqrt{C^2-3K_2D}}{D},\quad \widetilde{A}_0=-\frac{2\sqrt{C^2-3K_2D}+C}{3D},$$

$$\widetilde{A}_1=\frac{\sqrt{C^2-3K_2D}}{D}.$$

If we take the condition (18) into account, we conclude that for the existence of soliton solutions (24) and (25), the parameters k_1, K_2, a_0, l_0, and v and coefficients of Eq. (1) must satisfy the conditions

$$0<C^2-3K_2D,\quad 5Q_3^2+2Q_2Q_3-16PQ_1-3Q_2^2=0,$$

and

$$0=-\frac{108K_1^2D^2}{P^2}+9K_2D(-C+\sqrt{C^2-3K_2D})$$
$$+3C(-C+\sqrt{C^2-3K_2D})^2+(-C+\sqrt{C^2-3K_2D})^3.$$

The condition of positivity is also required on these solutions, and this imposes conditions on the domain of the above parameters. Some soliton solutions are shown on Figs. 5–7. We will use the values of parameters P and Q_j ($j=1,2,3$) given by Eq. (23). If we take $l_0=-2v-6a_0^2$ and

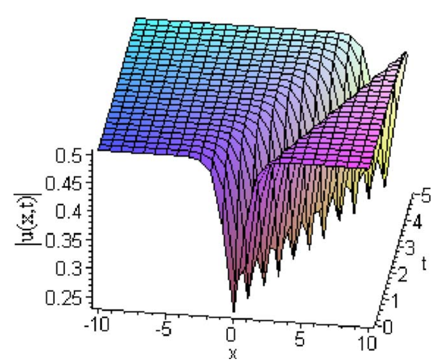

FIG. 6. (Color online) Evolution of type A $\mathrm{sn}(z,k)$ solution at limit $k\to 1$ for $v=-2^{3/4}$ over 5 time units.

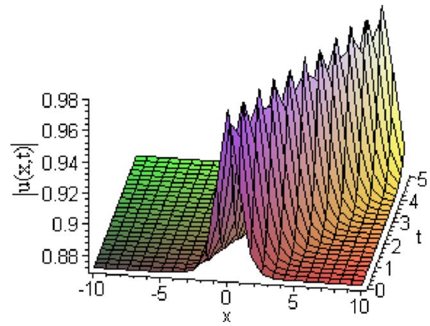

FIG. 7. (Color online) Evolution of type A $\mathrm{dn}(z,k)$ solution at limit $k \to 1$ for $v=-2^{3/4}$ over 5 time units.

$K_1=\frac{5}{8}a_0^4$, then we find that $D=-12v$ and $C=-4v^2$. For $K_2=(m^4-4)v^3/9$ for some real m, the first of the above conditions is valid, and the third one gives $a_0^8=[(64-16m^6-48m^4)/(27\times 36\times 25)]v^4$. By taking $m=(2)^{-1/4}$, the solutions (24) and (25) become

$$\zeta(z)=\frac{v}{6}\left(-\frac{1}{\sqrt{2}}-2+\frac{1}{\sqrt{2}\cosh^2[(2)^{-3/4}vz]}\right),$$

$$\zeta(z)=-\frac{v}{6}\left(\frac{1}{\sqrt{2}}+2+\frac{2}{\cosh^2[(2)^{-3/4}vz]}\right),$$

$$\zeta(z)=-\frac{v}{6}\left(\frac{(2-\sqrt{2})}{3}+\frac{\sqrt{2}}{2}\tanh^2[(2)^{-3/4}vz]\right),$$

respectively. These solutions are nonnegative as soon as $v<0$.

B. Type B: $E\neq 0$

Now, we consider Eq. (13) with $E\neq 0$. For this type, we seek bounded nonnegative solutions in one of the forms

$$\zeta(z)=A_0+A_1\mathrm{dn}(\mu z,k)$$

and

$$\zeta(z)=\widetilde{A}_0+\frac{\widetilde{A}_1}{\mathrm{dn}(\mu z,k)},$$

with $A_1\widetilde{A}_1\neq 0$. Inserting these expressions into Eq. (13) and equating different powers of $\mathrm{dn}(\mu z,k)$ imposes the following constraints on the parameters:

$$A_0=-\frac{D}{4E},\quad A_1^2=-\frac{\mu^2}{E},\quad \widetilde{A}_0=-\frac{D}{4E}, \tag{26}$$

$$\widetilde{A}_1^2=-\frac{\mu^2(1-k^2)}{E},\quad k^2=2+\frac{3D^2-8CE}{8E\mu^2}, \tag{27}$$

$$0=\mu^4+\frac{3D^2-8CE}{8E}\mu^2+\frac{P^2(CD^2-10K_2DE)-256K_1^2E^2}{64P^2E},$$

$$0=8K_2E^2-4CDE+D^3\quad\text{and}\quad E<0, \tag{28}$$

and

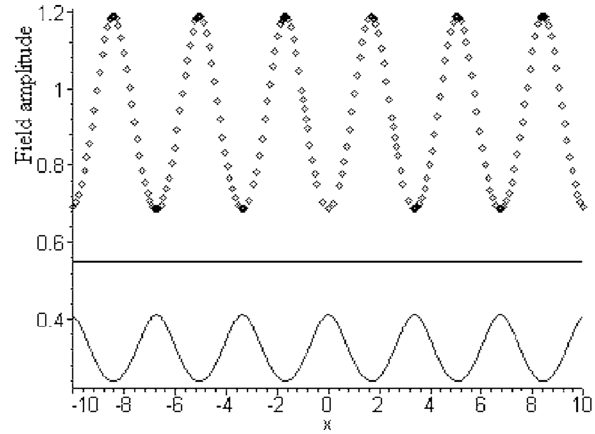

FIG. 8. Field amplitude of exact solution of type B $\mathrm{dn}(x,t)$ (solid line) and of type B $1/\mathrm{dn}(x,t)$ (dotted line) for $v=0$ and $k=0.5$.

$$0\leqslant 4P^2(3D^2-8CE)^2+1012K_1^2E^3+P^2D^2(16CE-5D^2). \tag{29}$$

$E<0$ is the condition of existence of A_1 and $\widetilde{A}_1$ given by Eqs. (26) and (27), respectively, while Eq. (29) is the condition of definition of μ^2. This condition is obtained using the first equation (28) where we express K_2 in terms of C, D, and E. We then obtain the following solutions:

$$\zeta_1^{\pm}(z)=-\frac{D}{4E}\pm\sqrt{-\frac{\mu^2}{E}}\mathrm{dn}(\mu z,k) \tag{30}$$

and

$$\zeta_2^{\pm}(z)=-\frac{D}{4E}\pm\sqrt{-\frac{\mu^2(1-k^2)}{E}}\frac{1}{\mathrm{dn}(\mu z,k)}. \tag{31}$$

The sign + or − is taken from the condition of the positivity of these solutions. Let us indicate the condition under which both solutions (30) and (31) are nonnegative. First of all, we notice that $\mathrm{dn}(z,k)$ lies in the segment $[\sqrt{1-k^2},1]$. Using this remark, one finds that $\zeta_j^+(z)$ will be nonnegative if and only if $D\geqslant -4\sqrt{(k^2-1)\mu^2E}$. For the nonnegativity of $\zeta_j^-(z)$, it is necessary that $D>0$ and sufficient that $D\leqslant 4\sqrt{-E\mu^2}$.

The region of validity of solutions (30) and (31) is once more determined by the requirements $0\leqslant k\leqslant 1$. We notice that the conditions of definition and the conditions of nonnegativity of solutions (30) and (31) are the same so that the existence and nonnegativity of one of these solutions implies the existence and nonnegativity of the other.

The graphical representation of solutions of type B $\mathrm{dn}(x,t)$ (solid line) and of type B $1/\mathrm{dn}(x,t)$ (dotted line) for $k=0.5$ and $v=0$ for different arguments x are shown in Fig. 8. The graphical representations of these figures have mirror symmetry; they have the same period but inverse phases. On this figure, the horizontal solid line represents the mirror.

If the conditions

$$E<0 \quad \text{and} \quad \frac{8}{3}CE<D^2<\frac{16}{5}CE$$

are satisfied, then at the limit $k\to 1$, one obtains from solution (30) the following soliton solution:

$$\zeta^{\pm}_{\lim}(z) = -\frac{D}{4E} \pm \sqrt{\frac{3D^2-8CE}{8E^2}}\frac{1}{\cosh(\mu z)}, \tag{32}$$

where $\mu^2=(8CE-3D^2)/8E$. We notice that in the limit $k\to 1$, we have

$$K_1^2 \to \frac{P^2D^2(5D^2-16CE)}{1012E^3}.$$

The sign + or − in Eq. (32) is taken from the condition of nonnegativity of $\zeta^{\pm}_{\lim}(z)$.

We now construct another pair of solutions having some other symmetry. Let us consider the special case when $D=0$ and let us take the constant of integration $K_2=0$. Then Eq. (13) becomes

$$\zeta'^2 = -\frac{4K_1^2}{P^2}+C\zeta^2+E\zeta^4. \tag{33}$$

The required solutions are found by substituting either

$$\zeta(z) = \frac{A_0\mathrm{dn}(\mu z,k)}{1+A_1\mathrm{sn}(\mu z,k)} \tag{34}$$

or

$$\zeta(z) = \frac{1+\widetilde{A}_0\mathrm{sn}(\mu z,k)}{\widetilde{A}_1\mathrm{dn}(\mu z,k)} \tag{35}$$

in Eq. (33). Here A_0, A_1, $\widetilde{A}_0$, and $\widetilde{A}_1$ are four constants so that A_1 and $\widetilde{A}_0$ lie in interval $(-1,1)$ and A_0 and $\widetilde{A}_1$ are positive, $0\leqslant k\leqslant 1$, and μ is a real number. Substituting Eq. (34) and the Eq. (35) in Eq. (33) and using some properties of Jacobian elliptic functions, we find

$$A_0^2=\frac{2K_1^2(C\pm 4|K_1P^{-1}|\sqrt{-E})}{-4K_1^2E\pm C|K_1P|\sqrt{-E}},$$

$$A_1^2=\frac{\pm 2(16K_1^2E+P^2C^2)|K_1P^{-1}|\sqrt{-E}}{(-4K_1^2E\pm C|K_1P|\sqrt{-E})(C\pm 4|K_1P^{-1}|\sqrt{-E})}, \tag{36}$$

$$\widetilde{A}_0^2=\frac{\mp(16K_1^2E+P^2C^2)|K_1P^{-1}|}{(C\pm 4|K_1P^{-1}|\sqrt{-E})(4K_1^2\sqrt{-E}\mp C|K_1P|)},$$

$$\widetilde{A}_1^2=-\frac{(-P^2C\sqrt{-E}\pm 4|K_1P|E)}{8K_1^2\sqrt{-E}\pm 2C|K_1P|}, \tag{37}$$

$$\mu^2=C\pm 4|K_1P^{-1}|\sqrt{-E},$$

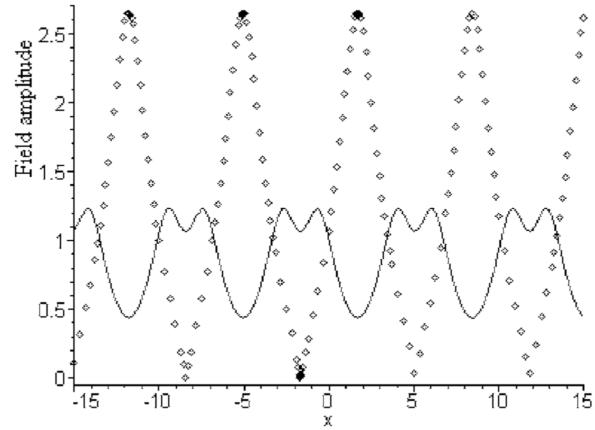

FIG. 9. Field amplitude of stationary solution ($v=0$) of type B $\mathrm{dn}(x,t)/[1+\mathrm{sn}(x,t)]$ (solid line) and of type B $\mathrm{sn}(x,t)/[1+\mathrm{dn}(x,t)]$ (dotted line) for $t=0$ over $x\in[-15,15]$.

$$k^2=\frac{P^2C^2+16K_1^2E}{(PC\pm 4|K_1|\sqrt{-E})^2}. \tag{38}$$

As we see from these equations, E must be negative and the second member of each equality must be positive, and this requirement restricts the region of validity of solution (34) and/or solution (35). In Eqs. (36)–(38) the sign + or − is taken from the condition that

$$\mu^2=C\pm 4|K_1P^{-1}|\sqrt{-E}>0.$$

It follows from Eqs. (10) that the requirement $D=0$ corresponds either to the stationary solution ($v=0$) or to Eq. (1) in which $Q_2=Q_3$. Some solutions are shown in Figs. 9 for the following coefficients of Eq. (1): $P=1$, $Q_2=Q_3=1$, $Q_1=127/400$. For the values of coefficients of Eq. (1) given in Eq. (23), we choose parameters v, a_0, and l_0 from the condition

$$127a_0^4-400v^2-1600(l_0^2+l_0v)=0,$$

and take $K_1=-0.3$. Figure 9 shows that the graphical representations of the two solutions show a kind of rotational symmetry, have the same period, different amplitudes, the same phase on one semiperiod, but 180° rotational phase difference on the next semiperiod.

IV. CONCLUSION

In this work we consider a derivative nonlinear Schrödinger equation that can be derived from a discrete nonlinear transmission line by using the reductive perturbation method and the continuum approximation method. We show that the equation under consideration admits Stokes wave solutions that can be perturbed to obtain exact soliton solutions and also elliptic solutions. This technique of seeking solutions of nonlinear equations increases the number of exact solutions that one can find, because of the number of parameters that figure in Stokes wave solutions, namely, the amplitude a_0 and the wave number l_0. The exact solutions are obtained when the values of the coefficients of the DNLS equation are related by certain equations. For some values of the coefficients we have found that the graphical representations of some exact elliptic solutions show some kinds of symmetries, such as mirror symmetry and rotational symmetry. Of course, more solutions of nonlinear partial differential equations may not take these types of symmetries.

ACKNOWLEDGMENTS

E.K. would like to acknowledge the CAS-TWAS for financial support. This work was supported by the NSF of China under Grants No. 60490280, No. 90403034, and No. 90406017, and the National Key Basic Research Special Foundation of China under Grant No. 2005CB724508.

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